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Existence of Optimal Control for LQ Optimization Problem Subject to Differential Algebraic Systems

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ABSTRACT

This paper deals with the LQ (Linear Quadratic) optimization problem subject to differential algebraic systems. The main aim of this paper is to find the optimal control-state pair for such problem. By transforming the differential algebraic system into a Weierstrass-Kronecker canonical reprentation, the LQ optimization problem can be classified into two types, i.e. the LQ optimization problem subject to proper systems and the LQ optimization problem subject to nonproper systems. The process of finding the solution of each problem is treated separately. By a certain condition, the singular LQ problem for both proper and nonproper systems are transformed into the standard LQ optimization problem. We solve both LQ optimization problems using classical optimal control theory.

Kata Kunci: *Differential algebraic system*, LQ *optimization*, *Weierstrass-Kronecker canonical*

I. INTRODUCTION

We consider the problem of minimizing the performance index

$$J(u,x)=\int_0^{t_1}\begin{pmatrix}x\\u\end{pmatrix}^T\cdot\begin{pmatrix}C^TC & C^TD\\D^TC & D^TD\end{pmatrix}\begin{pmatrix}x\\u\end{pmatrix}dt \qquad (1)$$

subject to the following differential algebraic system

$$\begin{aligned}E\dot{x}(t)&=Ax(t)+Bu(t),\quad Ex(0)=x_0,\ 0\le t\le t_1\\ y(t)&=Cx(t)+Du(t)\end{aligned} \qquad (2)$$

where $x\in\mathbb{R}^n$ denotes the state variable, $u\in\mathbb{R}^r$ denotes the control (input) variable, and $y\in\mathbb{R}^q$ denotes the output variable. This problem is often well known as the LQ problem subject to differential algebraic system [2-5]. Here the matrices $E, A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times r}$, $C\in\mathbb{R}^{q\times n}$, $D\in\mathbb{R}^{q\times r}$ are time invariant, $\text{rank}(E)=p<n$, and

$$\begin{pmatrix}C^TC & C^TD\\D^TC & D^TD\end{pmatrix}\ge 0.$$

For simplicity, denote the above LQ optimization problem as Π.

A fundamental question need to be answered concerning the existence and uniqueness of the optimal control-state pair (x^*,u^*) for the problem Π, i.e. under what conditions does there exist uniquely the optimal control-state pair (x^*,u^*)?. For the case in which $D^TD>0$ and $\det(sE-A)\neq 0$ for some $s\in\mathbb{R}$, has been already answered exhaustively in [2,5,6,7]. In the papers [4,5], Muller distinguishes between properness and non properness of the differential algebraic system.

To the best of the author's knowledge, not much work has been reported for the case $D^TD\ge 0$, thus it has remained open and motivates our present work. Nonetheless, the paper from Zhu *et al* [3] is of interest to consider. It solves the case in which the differential algebraic system need not regular and the output vector does not depend on input, using the singular value decomposition approach. In the present paper the problem to be solved consist of constructing an optimal pair (x^*, u^*) which minimizes (2) having regard to the dynamic system (1), for the case in which $D^T D \geq 0$ and $\det(sE - A) \neq 0$ for some $s \in \mathbb{R}$. Unlike [3-4], here we use Weierstrass-Kronecker canonical representation approach for the differential algebraic system (2).

2. TRANSFORMASI OF THE PROBLEM

To discuss the solution of Π we use the Weierstrass-Kronecker canonical representation for the differential algebraic system (2) [5-7]. It is well known that under condition $\det(sE - A) \neq 0$, there exist the nonsingular matrices $L, M \in \mathbb{R}^{n \times n}$ such that

$$LEM = \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix}.$$

Accordingly, let

$$LAM = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-p} \end{pmatrix},\ LB = LB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ CM = \begin{pmatrix} C_1 & C_2 \end{pmatrix},\ M^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3}$$

where $A_1 \in \mathbb{R}^{p \times p}$, $B_1 \in \mathbb{R}^{p \times r}$, $B_2 \in \mathbb{R}^{(n-p) \times r}$, $C_1 \in \mathbb{R}^{q \times p}$, $C_2 \in \mathbb{R}^{q \times (n-p)}$, $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$. N is a nilpotent matrix of index j (i.e. $N^j = 0$, $N^{j-1} \neq 0$) defining the index of the differential algebraic system. By the transformation (3), the performance index (1) can be rewritten as

$$J(u, x_1, x_2) = \int_0^{t_f} \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix}^T \begin{pmatrix} C_1^T C_1 & C_1^T C_2 & C_1^T D \\ C_2^T C_1 & C_2^T C_2 & C_2^T D \\ D^T C_1 & D^T C_2 & D^T D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} dt, \tag{4}$$

and the differential algebraic system (2) becomes:

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t),\ x_1(0) = x_{10} \\ N \dot{x}_2(t) &= x_2(t) + B_2 u(t) \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) + D u(t) \end{aligned} \tag{5}$$

where $x_{10} = (I_p \quad 0) L x_0$. Now we must minimize (4) subject to the system (5). The solution of the first differential equation of the system (5) is easily obtained in the classical manner, meanwhile the solution of the second equation of (5) is given

$$x_2(t) = -\sum_{i=0}^{k-1} N^i B_2 u^{(i)}(t) \tag{6}$$

where $u^{(i)}(t)$ denotes i^{th} derivative of $u(t)$. One can see that x_2 generally depends on the higher order time derivatives of control which is a very unusual behavior and must be regarded very carefully. In the following we distinguish the two cases where the solution depends either only on $u(t)$ but not on its derivatives $\dot{u}, \ddot{u}, \ldots, u^{(k-1)}$ or on $u(t)$ and its derivatives $\dot{u}, \ddot{u}, \ldots, u^{(k-1)}$.

Definition 2.1 *System (1) is termed as proper if its solution depends only on $u(t)$ but not on its derivatives $\dot{u}, \ddot{u}, \cdots, u^{(k-1)}$. Otherwise the system is nonproper.*

Base on definition 2.1, a criterion for properness is derived immediately.

Lemma 2.1 *The differential algebraic system (2) is proper if and only if $NB_2 = 0$ hold.*

The optimization problem of Π has to be performed in accordance to the properness and non properness of the differential algebraic system and so two different optimization problems have to be considered.

3. LQ OPTIMIZATION PROBLEM SUBJECT TO PROPER DIFFERENTIAL ALGEBRAIC SYSTEM

If $NB_2 = 0$ then the solution of the second equation of (5) is

$$x_2(t) = -B_2 u(t). \tag{7}$$

Creating the transformation

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & -B_2 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} x_1(t) \\ u(t) \end{pmatrix} \tag{8}$$

and replace (8) into the performance index (4) and the system (5), we have the standard LQ optimization problem which minimize the performance index

$$J(u, x_1) = \int_0^{t_1} \begin{pmatrix} x_1 \\ u \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} dt \tag{9}$$

subject to

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u, \ x_1(0) = x_{10} \\ y &= C_1 x_1 + (-C_2 B_2 + D) u, \end{aligned} \tag{10}$$

where $Q_{11} = C_1^T C_1$, $Q_{12} = C_1^T (D - C_2 B_2)$, $Q_{22} = (D - C_2 B_2)^T (D - C_2 B_2)$. Denote this LQ problem as Π_1. According to the linear optimal control theory for standard state space system [1], the problem has a unique solution if the system (10) is controllable and the matrix Q_{22} is positive definite.

Theorem 3.1 *Q_{22} is positive definite if and only if* rank $(C \quad D) = r$.

If the theorem 3.1 holds and the system (10) is controllable, one can see that Π_1 has a unique optimal control-state pair (u^*, x_1^*):

$$u^* = -K x_1^* \tag{11}$$

where x_1^* satisfies

$$\dot{x}_1 = (A_1 - B_1 K) x_1, \ x_1(0) = x_{10}, \tag{12}$$

with $K = Q_{22}^{-1}(Q_{12}^T + B_1^T P)$, P is a unique positive definite solution of Riccati differential equation

$$\dot{P} = -P(A_1 - B_1 Q_{22}^{-1} Q_{12}^T) - (A_1 - B_1 Q_{22}^{-1} Q_{12}^T)^T P + P B_1 Q_{22}^{-1} B_1^T P - Q_{11} + Q_{12} Q_{22}^{-1} Q_{12}^T \tag{13}$$

with the terminal condition $P(t_1) = 0$.

Theorem 3.2 *If $NB_2 = 0$ and* rank$(C \quad D) = r$, *then the LQ optimization problem Π has a unique optimal control-state pair.*

Proof. Let the hypothesis hold, then the optimal solution of Π_1 is given by (11)-(13). Substituting (11) into (8) and using (3) then the unique optimal control-state pair (u^*, x^*) of Π is

$$\begin{pmatrix} x^* \\ u^* \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ \hline u^* \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & I_r \end{pmatrix} \left(\begin{array}{c|c} I_p & 0 \\ 0 & -B_2 \\ \hline 0 & I_r \end{array} \right) \begin{pmatrix} I_p \\ -K \end{pmatrix} x_1^* = \begin{pmatrix} M & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} 0 \\ -B_2 K \\ \hline -K \end{pmatrix} x_1^*$$

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4. LQ OPTIMIZATION PROBLEM SUBJECT TO NONPROPER DIFFERENTIAL ALGEBRAIC SYSTEM

For the nonproper problem we need an extension of state and control variables to deal correctly with the influence of the time derivatives of the control input. Let

$$v_1 = u, v_2 = \dot{u}, v_3 = \ddot{u}, \cdots, v_{k-1} = u^{(k-2)}, w = u^{(k-1)}, \tag{14}$$

then (6) is replaced by

$$x_2 = -B_2 v_1 - NB_2 v_2 - N^2 B_2 v_3 - \cdots - N^{k-2} B_2 v_{k-1} - N^{k-1} B_2 w = -(0 \quad \overline{B}_2) x_e - N^{k-1} B_2 w \tag{15}$$

where

$$x_e = (x_1^T \quad v_1^T \quad v_2^T \cdots v_{k-1}^T)^T,$$

and

$$\overline{B}_2 = (B_2 \quad NB_2 \quad N^2 B_2 \cdots N^{k-2} B_2).$$

Here w is-considered as a new control vector. Create the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = \left(\begin{array}{ccccc|c} I_n & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & -B_2 & -NB_2 & \cdots & -N^{k-2}B_2 & -N^{k-1}B_2 \\ \hline 0 & I_r & 0 & \cdots & 0 & 0 \end{array}\right) \begin{pmatrix} x_e \\ w \end{pmatrix} = \left(\begin{array}{cc|c} I_n & 0 & 0 \\ \hline 0 & -\overline{B}_2 & -N^{k-1}B_2 \\ \hline 0 & I_{r,0} & 0 \end{array}\right) \begin{pmatrix} x_e \\ w \end{pmatrix}, \tag{16}$$

where $I_{r,0} = (I_r \quad 0 \quad \cdots \quad 0)$. Substituting (16) into the performance index (4), we have

$$J(w, x_e) = \int_0^{\infty} \begin{pmatrix} x_e \\ w \end{pmatrix}^T \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{12}^T & \overline{Q}_{22} \end{pmatrix} \begin{pmatrix} x_e \\ w \end{pmatrix} dt, \tag{17}$$

where

$$\overline{Q}_{11} = \left(\begin{array}{c|c} C_1^T C_1 & -C_1^T C_2 \overline{B}_2 + C_1^T D I_{r,0} \\ \hline -\overline{B}_2^T C_2^T C_1 + I_{r,0}^T D^T C_1 & (C_2\overline{B}_2)^T C_2 \overline{B}_2 - (DI_{r,0})^T C_2 \overline{B}_2 - (C_2 \overline{B}_2)^T DI_{r,0} + (DI_{r,0})^T DI_{r,0} \end{array}\right),$$

$$\overline{Q}_{12} = \begin{pmatrix} -C_1^T C_2 N^{k-1} B_2 \\ (C_2 \overline{B}_2)^T C_2 N^{k-1} B_2 - (DI_{r,0})^T C_2 N^{k-1} B_2 \end{pmatrix},$$

and

$$\overline{Q}_{22} = \left(C_2 N^{k-1} B_2\right)^T \left(C_2 N^{k-1} B_2\right).$$

Meanwhile, using (16) the first and the third equations of the system (5) can be extended as

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e w(t), \quad x_e(0) = x_{e0} \\ y(t) &= \overline{C} x_e(t) + (-C_2 N^{k-1} B_2) w(t) \end{aligned} \tag{18}$$

where

$$A_e = \begin{pmatrix} A_1 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I_r \end{pmatrix} \quad \text{and} \quad \overline{C} = \left(C_1 \quad -C_2 \overline{B}_2 + DI_{r,0}\right).$$

Now we arrive at the new standard LQ optimization problem which minimize (17) subject to (18). Actually, when $\text{rank}\,(C_2 N^{k-1} B_2) = r$ then we will have $\overline{Q}_{22} > 0$, but in fact, since N is a nilpotent matrix then $\text{rank}\,(C_2 N^{k-1} B_2) < r$. Thereby it is obvious that we arrive at a standard singular LQ problem with respect to variables x_e and w. Henceforth, the two kinds singular LQ problems must be solved, i.e. the cases in which $\overline{Q}_{22} = 0$ (totally singular) and. $\overline{Q}_{22} \geq 0$ (partially singular). Here we only restrict the

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discussion for the case $\overline{Q}_{22} = 0$ (totally singular), meanwhile for the case $\overline{Q}_{22} \geq 0$ (partially singular) will be discussed in the future work. By the assumption $C_2 N^{k-1} B_2 = 0$, we will solve the totally singular LQ optimization problem with performance index

$$J(w, x_e) = \int_0^{t_1} x_e^T \overline{Q}_{11} x_e \, dt. \tag{19}$$

Therefore, taking the following transformation:

$$z(t) = x_e(t) - B_e \eta(t), \quad \text{and} \quad \eta(t) = \int_0^t w(\tau) d\tau, \tag{20}$$

and substituting (20) into (18) and (19), we have the LQ optimization problem which minimize

$$J(\eta, z) = \int_0^{t_1} (z + B_e \eta)^T \overline{Q}_{11} (z + B_e \eta) dt = \int_0^{t_1} \begin{pmatrix} z \\ \eta \end{pmatrix}^T \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{11} B_e \\ (\overline{Q}_{11} B_e)^T & B_e \overline{Q}_{11} B_e \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix} dt \tag{21}$$

subject to

$$\begin{aligned} \dot{z} &= A_e z + A_e B_e \eta, \; z(0) = x_{e0} - B_e \eta_0 \\ y &= \overline{C} z + \overline{C} B_e \eta \end{aligned} \tag{22}$$

and denote this LQ problem as Π_2. One can see that when $B_e \overline{Q}_{11} B_e$ is positive definite and the system (22) is controllable then with the tool of linear optimal control theory, Π_2 has a unique optimal control-state pair (η^*, z^*):

$$\eta^* = -\overline{K} z^* \tag{23}$$

where z^* satisfies

$$\dot{z} = (A_e - A_e B_e \overline{K}) z, \quad z(0) = x_{e0} - B_e \eta_0 \tag{24}$$

with $\overline{K} = (B_e \overline{Q}_{11} B_e)^{-1} (B_e^T \overline{Q}_{11}^T + B_e^T A_e^T \overline{P})$, $\overline{P}$ is the unique positive definite solution of Riccati differential equation

$$\begin{aligned} \dot{\overline{P}} = &-\overline{P}(A_e - A_e B_e (B_e \overline{Q}_{11} B_e)^{-1} (\overline{Q}_{11} B_e)^T) - (A_e - A_e B_e (B_e \overline{Q}_{11} B_e)^{-1} (\overline{Q}_{11} B_e)^T)^T \overline{P} \\ &+ \overline{P} A_e B_e (B_e \overline{Q}_{11} B_e)^{-1} (A_e B_e)^T \overline{P} - \overline{Q}_{11} + \overline{Q}_{11} B_e (B_e \overline{Q}_{11} B_e)^{-1} (\overline{Q}_{11} B_e)^T \end{aligned} \tag{25}$$

with the terminal condition $\overline{P}(t_1) = 0$.

Theorem 4.1 *If* $C_2 N^{k-1} B_2 = 0$ *then the LQ optimization problem* Π *has a unique optimal control-state pair.*

Proof. When the hypothesis holds then the optimal solution of Π_2 is given by (23)-(25). Base on (20) and (23), we have

$$\begin{pmatrix} x_e^* \\ w^* \end{pmatrix} = \begin{pmatrix} I_{p+(k-1)r} + B_e \overline{K} & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} z^* \\ w^* \end{pmatrix}, \tag{26}$$

and substituting (26) into (16) we obtain

$$\begin{pmatrix} x_1^* \\ x_2^* \\ u^* \end{pmatrix} = \left(\begin{array}{cc|c} I_p & 0 & 0 \\ \hline 0 & -\overline{B}_2 & -N^{k-1} B_2 \\ \hline 0 & I_{r,0} & 0 \end{array}\right) \left(\begin{array}{c|c} I_{p+(k-1)r} + B_e \overline{K} & 0 \\ \hline 0 & I_r \end{array}\right) \begin{pmatrix} z^* \\ w^* \end{pmatrix}. \tag{27}$$

Since $x = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then the unique optimal control-state pair (u^*, x^*) of Π is

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$$\begin{pmatrix} x^* \\ u^* \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ \hline u^* \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & I_r \end{pmatrix} \left(\begin{array}{cc|c} I_p & 0 & 0 \\ \hline 0 & -\overline{B}_2 & -N^{k-1}B_2 \\ \hline 0 & I_{r,p} & 0 \end{array} \right) \begin{pmatrix} I_{p+(k-1)r} + B_e\overline{K} & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} z^* \\ w^* \end{pmatrix} . \square$$

5. CONCLUSION

We have solved the singular LQ optimization problem for differential algebraic system and obtain the explicit form of the control optimal-state pair of the problem. How to solve the LQ optimization problem for nonproper differential algebraic system of the partially singular case is an open question.

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